RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2017

FIRST YEAR [BATCH 2017-20]

MATHEMATICS (Honours)

Date : 12-12-2017

Time : 11.00 am – 3.00 pm

Paper : I

Full Marks : 100

(Use a separate Answer book for each group)

Group – A

Answer **any five** questions from ***Question Nos.1 to Question Nos. 8*** . 5 X 5

1. H and K are different subgroups of a group G such that $O(H) = O(K) = p$, where p is prime. Show that $H \bigcap K = \{e\}$. Deduce that if G has exactly m distinct subgroups of order p, then the total number of elements of order p in G is $m(p-1)$. 3+2

2. Suppose H and K are subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$. 5

3. (i) In a group $(G, \circ)$, the elements a and b commute and $O(a)$, $O(b)$ are prime to each other. Show that $O(a,b) = O(a) \cdot O(b)$.
   (ii) Find all elements of order 8 in the group $(Z_{24}, +)$. 3+2

4. If $\alpha = (134)\ (56)\ (2789)$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$ are two permutations on $\{1, 2, \cdots\cdots, 9\}$, find $\gamma = \beta^{-1}\alpha\beta$. Is γ an even permutation? Justify whether γ can be expressed as a product of transpositions uniquely. 2+1+2

5. Show that the number of different reflexive relations on a set of n elements is 2^{n^2-n}. 5

6. A map $f : S \to$ □ is defined by $f(x) = \dfrac{x}{1-|x|}$, where $S = \{x \in$ □ $: -1 < x < 1\}$. Prove that f is a bijection. Hence find f^{-1}. 3+2

7. Let $f : A \to B$ be a function. Prove that
   (i) if f is left invertible then f is injective.
   and (ii) if f is right invertible then f is surjective. 3+2

8. Give an example of a map from □ to □ which is surjective but not injective. Prove that every group of prime order is cyclic. 3+2

Answer **any five** questions from ***Question Nos. 9 to Question Nos. 16*** . 5 X 5

9. If $x, y \in \mathbb{R}$ with $y > 0$, then prove that $\exists n \in \mathbb{N}$ such that $ny > x$. Using Archimedean property, show that if a is any real number then $\lim_{n\to\infty} \frac{a}{n} = 0$. 3+2

10. Prove that the union of a finite number of open sets in $\mathbb{R}$ is open. Is it true for an arbitrary collection of open sets in $\mathbb{R}$? 4+1

11. (i) Prove that $A = \{x \in \mathbb{R} : \sin x \neq 0\}$ is an open set.

    (ii) For any two subsets A and B of $\mathbb{R}$, prove that $(A \cup B)' = A' \cup B'$ where S' denotes the derived set of S. 2+3

12. Prove that each interior point of a set is a limit point of the set but the converse is not true. 5

13. (i) If $\{a_n\}$ and $\{b_n\}$ are two real sequences such that $\lim a_n = 0$ and $\{b_n\}$ is bounded then prove that $\lim(a_n b_n) = 0$.

    (ii) Prove or disprove: Every bounded sequence is convergent. 3+2

14. State Cauchy's general principle of convergence of a sequence $\{u_n\}_n$.

    Using Cauchy's criterion of convergence, examine the convergence of the sequence $\{u_n\}_n$ where $u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. 2+3

15. If for a sequence $\{x_n\}$ of real numbers, $\lim_{n\to\infty} x_{3n-2} = \lim_{n\to\infty} x_{3n-1} = \lim_{n\to\infty} x_{3n} = \ell (\in \mathbb{R})$ then prove that $\{x_n\}$ is convergent. Find its limit. 5

16. Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$. Let $a \in S'$, the derived set of S. If $\lim_{x\to a} f(x) = \ell$, prove that $\lim_{x\to a} |f(x)| = |\ell|$. Give an example to show that the converse is not true. 4+1

Group – B

Answer **any five** questions from ***Question Nos. 17 to Question Nos. 24*** . 5×5

17. Show that the equation of the line joining the feet of perpendiculars from the point $(d, 0)$ on the lines $ax^2 + 2hxy + by^2 = 0$ is $(a-b)x + 2hy + bd = 0$. 5

18. Find the condition that the straight line $\frac{l}{r} = a\cos\theta + b\sin\theta$ may touch the circle $r = 4c\cos\theta$. 5

19. Show that the conics $ax^2 + 2hxy + by^2 + 2gx\cos^2\alpha + 2fy\sin^2\alpha + c = 0$, where α is a parameter, always pass through two fixed points, provided $g^2f^2 > c(af^2 + 2hfg + bg^2)$. 5

20. Show that the locus of the pole of the line joining the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$. 5

21. Show by vector method that the points $P(1,5,-1), Q(0,4,5), R(-1,5,1)$ and $S(2,4,3)$ are coplanar. 5

22. By vector method prove that $\sin(A-B) = \sin A\cos B - \cos A\sin B$. 5

23. Solve the vector equation $t\vec{r} + \vec{r}\times\vec{a} = \vec{b}$ where $\vec{a}$ and $\vec{b}$ are given vectors and t is a given scalar. 5

24. Find the shortest distance between the lines determined by the equations $3x - 4y - z + 5 = 0 = 3x - 6y - 2z + 13$ and $3x + 4y - 3z + 2 = 0 = 3x - 2y + 6z + 17$. 5

Answer **any five** questions from ***Question Nos. 25 to Question Nos. 32*** **.** 5×5

25. Obtain the differential equation of the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where λ is an arbitrary parameter and *a*, *b* are given constants. 5

26. Find an integrating factor of the differential equation $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ and using this integrating factor solve this equation. 2+3

27. Show that if y_1 and y_2 are solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone and $y_2 = y_1 z$, then $z = 1 + a\,e^{-\int \frac{Q}{y_1}dx}$, a being an arbitrary constant. 5

28. Solve the following differential equation by the method of variation of parameters:. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$. 5

29. Reduce the differential equation $y^2(y - px) = x^4p^2$ to Clairaut's form by the substitution $u = \frac{1}{x}$ and $v = \frac{1}{y}$ and find its complete primitive and its singular solution, if any. 2+3

30. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, λ being the parameter. 5

31. Solve the following differential equation using the method of undetermined coefficients:

$\frac{d^2y}{dx^2}-9y=x+e^{2x}-\sin 2x$. 5

32. Solve the following differential equation by transforming it to a linear equation with constant coefficients: $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$. 5

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